1. Let $A = \begin{pmatrix} -2 & 1 & 1 & 4 \\ 4 & -2 & -1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 14 \\ 4 \end{pmatrix}$.

   (a) Find the RREF for A.

   (b) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

   (c) What is the rank of A?

   (d) Find the dimensions of the four fundamental subspaces for A.

   (e) Find a basis for the null space of A.

   (f) Find an orthogonal basis for the null space of A. (It does not need to be orthonormal.)

   (g) Find the projection of $(1, 0, 0, 0)$ onto $N(A)$.

   (h) (Optional, 0 pts) Find the projection of $(1, 0, 0, 0)$ onto $C(A^T)$.

Solution:

(a) Using the row operations $R2 + 2R1$, then $R1 - R2$, then $R1/(-2)$, we get the RREF for A is $\begin{pmatrix} 1 & -1/2 & 0 & 2 \\ 0 & 0 & 1 & 8 \end{pmatrix}$.

(b) All solutions to $A\mathbf{x} = \mathbf{b}$ are

$$\{(9, 0, 32, 0) + c(1/2, 1, 0, 0) + d(-2, 0, -8, 1) : c, d \in \mathbb{R}\}$$

(c) Since R has two pivots, the rank of A is 2.

(d) Since the rank of A is 2, and A is 2×4, by the Rank-nullity Theorem,

- $\dim C(A) = 2$
- $\dim C(A^T) = 2$
- $\dim N(A) = 2$
- $\dim N(A^T) = 0$

(e) A basis for the null space of A is given by the columns of the matrix $N = \begin{pmatrix} 1/2 & -2 \\ 1 & 0 \\ 0 & -8 \\ 0 & 1 \end{pmatrix}$.

(f) We do Gram-Schmidt on the basis $\{\mathbf{v}, \mathbf{w}\} = \{(1/2, 1, 0, 0), (-2, 0, -8, 1)\}$. We take $\mathbf{v}$ as the first vector in the basis.
For the second vector, we take

$$\mathbf{w} - \text{proj}_{\mathbf{v}}\mathbf{w} = \mathbf{w} - \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \begin{pmatrix} -2 \\ 0 \\ -8 \\ 1 \end{pmatrix} - \begin{pmatrix} -2/5 \\ -4/5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -8/5 \\ 4/5 \\ -8 \\ 1 \end{pmatrix}$$

Thus an orthogonal basis is $\{(1/2, 1, 0, 0), (-8/5, 4/5, -8, 1)\}$

(g) The projection of $(1, 0, 0, 0)$ onto $N(A)$ can be calculated as the sum of the projections onto the orthogonal basis vectors.
The projections are $(1/5, 2/5, 0, 0)$ and $\frac{(-8/5)}{(-8/5,4/5,-8,1)^T(-8/5,4/5,-8,1)}(-8/5, 4/5, -8, 1)$, so the answer is the sum of these.

2. Suppose A is the product of the following elimination matrices.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

In this problem, you should not compute A, A^{-1}, or A^T explicitly.

(a) Write A^{-1} as a product of elimination matrices.

(b) Write A^T as a product of elimination matrices.

(c) Is A invertible? Explain how you know.

(d) What is the determinant of A?

(e) Does the linear transformation A preserve orientation, reverse orientation, or neither?

(f) Suppose R is a region of $\mathbb{R}^3$ with volume 10. Let $A(R)$ be the image of R after being transformed by A. What is the volume of $A(R)$?

(g) If $\mathbf{b} \in \mathbb{R}^3$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have?

(h) Find all solutions to $A\mathbf{x} = (1, 1, 1)$.

Solution:

(a) A^{-1} is the product of the inverses of each elimination matrices, in the opposite order. I.e.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) A^T is the product of the transposes of each elimination matrices, in the opposite order. I.e.

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(c) Since A is a product of invertible matrices, it is invertible.

(d) The determinant of A is the product of the determinants of the elimination matrices, so

$$\det A = (1)(1)(4)(1)(1) = 4.$$

(e) Since $\det A > 0$, A preserves orientation.

(f) The volume of $A(R)$ is scaled by the magnitude of the determinant, so it is $4 \cdot 10 = 40$.

(g) Since A is invertible, $A\mathbf{x} = \mathbf{b}$ has exactly one solution

(h) $\mathbf{x} = A^{-1}(1, 1, 1) = (-1. - 1/2, 1)$.

3. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^n$. For each of the following statements, either prove it or provide a counterexample.

   (a) $\text{span}\{\mathbf{v}, \mathbf{w}\} \subseteq \text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\}$

   (b) $\text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\} \subseteq \text{span}\{\mathbf{v}, \mathbf{w}\}$

   (c) $\text{proj}_{\mathbf{v}}\mathbf{w} = \text{proj}_{2\mathbf{v}}\mathbf{w}$

   (d) $\text{proj}_{\mathbf{v}}\mathbf{w} = \text{proj}_{\mathbf{v}}2\mathbf{w}$

   Solution:

   (a) Proof: Let $\mathbf{x} \in \text{span}\{\mathbf{v}, \mathbf{w}\}$. By the definition of span, there are real numbers a and b such that $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$. Thus $\mathbf{x} = a\mathbf{v} + b\mathbf{w} + 0(\mathbf{v} + \mathbf{w})$. By the definition of span, $\mathbf{x} \in \text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\}$. Thus, $\text{span}\{\mathbf{v}, \mathbf{w}\} \subseteq \text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\}$

   (b) Proof: Let $\mathbf{x} \in \text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\}$. By the definition of span, there are real numbers a, b, and c such that $\mathbf{x} = a\mathbf{v} + b\mathbf{w} + c(\mathbf{v} + \mathbf{w})$. Thus $\mathbf{x} = (a + c)\mathbf{v} + (b + c)\mathbf{w}$. By the definition of span, $\mathbf{x} \in \text{span}\{\mathbf{v}, \mathbf{w}\}$. Thus, $\text{span}\{\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\} \subseteq \text{span}\{\mathbf{v}, \mathbf{w}\}$

   (c) Proof: $\text{proj}_{2\mathbf{v}}\mathbf{w} = \frac{(2\mathbf{v})^T\mathbf{w}}{(2\mathbf{v})^T(2\mathbf{v})}(2\mathbf{v}) = \frac{2(\mathbf{v}^T\mathbf{w})}{4(\mathbf{v}^T\mathbf{v})}(2\mathbf{v}) = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{w}$

   (d) Since
   $$\text{proj}_{\mathbf{v}}2\mathbf{w} = \frac{\mathbf{v}^T(2\mathbf{w})}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = 2\frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = 2\text{proj}_{\mathbf{v}}\mathbf{w},$$
   the statement $\text{proj}_{\mathbf{v}}\mathbf{w} = \text{proj}_{\mathbf{v}}2\mathbf{w}$ is false unless $\mathbf{v}$ is orthogonal to $\mathbf{w}$.

4. Let $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ be a set of orthonormal vectors in $\mathbb{R}^5$, and let Q be the 5×3 matrix where the jth column of Q is equal to $\mathbf{q}_j$. Let V be the subspace of $\mathbb{R}^5$ spanned by $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$.

   (a) What is the dimension of V?

   (b) Find (in terms of Q, simplified) a matrix P such that for any $\mathbf{b} \in \mathbb{R}^5$, $P\mathbf{b}$ is the projection of $\mathbf{b}$ onto V.

   (c) Which fundamental subspace for Q is $V^\perp$? Explain.

   (d) Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$. Prove that $\|Q\mathbf{v}\| = \|\mathbf{v}\|$.

   (e) Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$. Explain why there must be vectors where $\|Q^T\mathbf{v}\| \neq \|\mathbf{v}\|$. Hint: Use the fact that Q^T has dependent columns.

   (f) Suppose $A = QR$, where R is an upper triangular matrix. Which of the following must be true?

      i. The first column of A is a scalar multiple of the first column of Q.
      ii. The first column of A is a scalar multiple of the first column of R.
      iii. The last column of A is a scalar multiple of the last column of Q.
      iv. The last column of A is a scalar multiple of the last column of R.

   (g) (Optional, 0 pts) Suppose $A = QR$, where R is an upper triangular matrix. Prove that if R has no zeros on the diagonal, then the columns of A are independent.

Solution:

(a) The dimension of V is 3.

(b) $P = Q(Q^TQ)^{-1}Q^T = QI^{-1}Q^T = QIQ^T = QQ^T$.

(c) $V^\perp$ is the orthogonal complement of $C(Q)$, so it is $N(Q^T)$, the left null space of Q.

(d) $\|Q\mathbf{v}\| = \sqrt{(Q\mathbf{v})^T(Q\mathbf{v})} = \sqrt{\mathbf{v}^TQ^TQ\mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = \|\mathbf{v}\|$.

(e) Since Q^T has more columns than rows (it is 3×5), it has dependent columns. Thisi means that there is a nonzero vector $\mathbf{v}$ in the null space of Q^T, i.e. $Q^T\mathbf{v} = \mathbf{0}$. In this case, $\|Q^T\mathbf{v}\| \neq \|\mathbf{v}\|$.

(f) The first column of A is a scalar multiple of the first column of Q. Specifically,

$$A_{*1} = R_{1,1}Q_{*1}.$$

The other statements are generally not true.

5. Suppose A and B are 3×3 matrices with the same eigenvectors. Suppose A has eigenvalues 5, 2, and 0, and B has eigenvalues 4, 3, and 3, for the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, respectively.

   (a) For each of the following matrices, if $\mathbf{x}_1$ is an eigenvector, state its eigenvalue. If it is not an eigenvector, explain how you know.
   - AB
   - $(A + B)$
   - B^7
   - B^{-1}
   - $(A + 2B^{-1})^7$

   (b) Prove or disprove: Any nonzero linear combination of $\mathbf{x}_2$ and $\mathbf{x}_3$ is an eigenvector for A.

   (c) Prove or disprove: Any nonzero linear combination of $\mathbf{x}_2$ and $\mathbf{x}_3$ is an eigenvector for B.

   (d) For each of the matrices A, B, AB and $A + B$, explain if they are diagonalizable, and describe a diagonalization if so. In each case where the diagonalization exists, write the matrix Λ explicitly.

Solution:

   (a) For each of the following matrices, $\mathbf{x}_1$ is an eigenvector, and we state its eigenvalue.
   - AB: $\lambda = 5 \cdot 4 = 20$
   - $(A + B)$: $\lambda = 5 + 4 = 9$
   - B^7: $\lambda = 4^7$
   - B^{-1}: $\lambda = 1/4$
   - $(A + 2B^{-1})^7$: $\lambda = (5 + 2 \cdot (1/4))^7 = 5.5^7$

   (b) This is false. Consider the nonzero linear combination of $\mathbf{x}_2 + \mathbf{x}_3$:

$$A(\mathbf{x}_2 + \mathbf{x}_3) = A\mathbf{x}_2 + A\mathbf{x}_3 = 2\mathbf{x}_2 + 0\mathbf{x}_3 = 2\mathbf{x}_2$$

   , which is not a scalar multiple of $\mathbf{x}_2 + \mathbf{x}_3$.

   (c) This is true. Consider an arbitrary nonzero linear combination of $\mathbf{x}_2 + \mathbf{x}_3$:

$$B(a\mathbf{x}_2 + b\mathbf{x}_3) = a(B\mathbf{x}_2) + b(B\mathbf{x}_3) = a(3\mathbf{x}_2) + b(3\mathbf{x}_3) = 3(a\mathbf{x}_2 + b\mathbf{x}_3),$$

   so $a\mathbf{x}_2 + b\mathbf{x}_3$ is an eigenvector with eigenvalue 3.

   (d) Let $X = \begin{pmatrix}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\end{pmatrix}$. Then
   - $A = X \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} X^{-1}$.
   - $B = X \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} X^{-1}$.
   - $AB = X \begin{pmatrix} 20 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} X^{-1}$.

- $A + B = X \begin{pmatrix} 9 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} X^{-1}$.

6. Suppose A has eigenvectors $\mathbf{x}_1 = (1, 0, 0)$, $\mathbf{x}_2 = (0, 1, 1)$, and $\mathbf{x}_3 = (0, 1, -1)$, with eigenvalues $\lambda_1 = 6$, $\lambda_2 = 2$, and $\lambda_3 = 0$. Let $\mathbf{v} = (3, 6, -2)$.

   (a) Show that the eigenvectors are orthogonal.

   (b) Write $\mathbf{v}$ as a linear combination of the eigenvectors.

   (c) Write $A^{33}\mathbf{v}$ as a linear combination of the eigenvectors.

   (d) Is A invertible? Explain how you know.

   (e) What is the determinant of A?

   (f) Does the linear transformation A preserve orientation, reverse orientation, or neither?

   (g) Find a basis for the column space of A.

   (h) Find a basis for the null space of A.

   (i) What is the rank of A?

   (j) Is A symmetric? If not, explain how you know. If so, find an orthogonal diagonalization.

Solution:

(a) Since the dot product of any pair is equal to 0, the eigenvectors are orthogonal.

(b) Since the eigenbasis is orthogonal, we can write $\mathbf{v}$ as a linear combination of the eigenvectors by taking the projection onto each one and summing. This gives

$$\mathbf{v} = 3\mathbf{x}_1 + 2\mathbf{x}_2 + 4\mathbf{x}_3.$$

(c)

$$\begin{aligned} A^{33}\mathbf{v} &= A^{33}(3\mathbf{x}_1 + 2\mathbf{x}_2 + 4\mathbf{x}_3) \\ &= 3A^{33}\mathbf{x}_1 + 2A^{33}\mathbf{x}_2 + 4A^{33}\mathbf{x}_3 \\ &= 3 \cdot 6^{33}\mathbf{x}_1 + 2 \cdot 2^{33}\mathbf{x}_2 + 4 \cdot 0^{33}\mathbf{x}_3 \\ &= 3 \cdot 6^{33}\mathbf{x}_1 + 2 \cdot 2^{33}\mathbf{x}_2 \end{aligned}$$

(d) Since the determinant is the product of the eigenvalues, $\det A = 0$, so A is not invertible.

(e) $\det A = 0$

(f) Since $\det A = 0$, A neither preserves nor reverses orientation.

(g) $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for the column space of A.

(h) $\{\mathbf{x}_3\}$ is a basis for the null space of A.

(i) The rank of A is 2.

(j) Since it has an eigenbasis of orthogonal eigenvectors, A is symmetric. To find an orthogonal diagonalization, we normalize the eigenvectors to be unit vectors:

$$A = Q\Lambda Q^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

7. Suppose A is a rank 2 matrix with rank 2 format singular value decomposition

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix}$$

(a) How many rows and columns does A have?

(b) What are the dimensions of the four fundamental subspaces of A?

(c) If $\mathbf{b} \in \mathbb{R}^3$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have?

(d) Find the rank 1 approximation A_1 for A given by the SVD. Your answer should be a single matrix.

(e) Is $C(A) \subseteq C(A_1)$, $C(A_1) \subseteq C(A)$, or neither?

(f) Find a vector $\mathbf{v}$ with length 3 such that $\|A\mathbf{v}\|$ is maximized. What vector is $A\mathbf{v}$?

(g) If $\mathbf{v}$ is a unit vector, what is the minimum length of $\|A\mathbf{v}\|$?

(h) What are the eigenvalues of A^TA?

(i) What are the eigenvalues of AA^T?

Solution:

(a) A is 3×4.

(b) Since A has rank 2,

- $\dim C(A) = 2$
- $\dim C(A^T) = 2$
- $\dim N(A) = 2$
- $\dim N(A^T) = 1$

(c) Since A has neither full row rank nor full column rank $A\mathbf{x} = \mathbf{b}$ could have 0 or infinite solutions.

(d)

$$A_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} 8 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) Since all columns of A_1 are multiples of $\mathbf{u}_1$, and all $C(A)$ is is the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$, $C(A_1) \subseteq C(A)$.

(f) A $\mathbf{v}$ with length 3 that maximizes $\|A\mathbf{v}\|$ is $3\mathbf{v}_1 = (3/2, 3/2, 3/2, 3/2)$. In this case $A\mathbf{v} = 3 \cdot 8\mathbf{u}_1 = (24/\sqrt{2}, 24/\sqrt{2}, 0)$.

(g) If $\mathbf{v}$ is a unit vector, the minimum length of $\|A\mathbf{v}\|$ is 0 (if $\mathbf{v}$ is any unit vector in the null space).

(h) The eigenvalues of A^TA are 64, 25, 0,0

(i) The eigenvalues of AA^T are 64, 25, 0